**Math 358 – Foundations of Mathematics** **Fall 2008**
**8-21-08 Class Notes**

**Sections 1.1, 1.2**
1.1: 1, 3, 7, 9
1.2: 18, 23, 24, 27

## Propositional Logic

We are going to spend considerable time proving things and going through arguments. Therefore, we need to have an agreement about the meanings to many of our words. Here they are:

**Definition:** A **proposition** (or **statement**) is a sentence (either mathematical or alphabetic) that is either true or false. The truth or falseness of a proposition is called its **truth value**.

**Example 1:** Which of the following are propositions?

(a) $1+1=2$.
(b) Texas is the 2nd largest state in the union.
(c) What time is it?
(d) Nicholls has the #1 ranked college basketball team in the country.
(e) $3+4=6$.
(f) Go to school.
(g) The three cube roots of 1 are 1, $\frac{-1+\sqrt{3}}{2}$, and $\frac{-1-\sqrt{3}}{2}$.
(h) $x^2-3=1$.
(i) This statement is false.

Statements (c), (f), (h), and (i) are not propositions because we cannot answer whether they are true or false. The rest are all propositions ((a), (b), and (g) are true while (d) and (e) are false). You might wonder why statement (h) is not a proposition. The reason is that it depends on the value of $x$. For some $x$-values it is true and for the rest it is false. However, we'll see later that it can be made into a proposition in two ways.

Given two propositions, how do we use them to produce new propositions?

**Definition:** If $p$ and $q$ are propositions, then their **conjunction**, denoted by $p \wedge q$, is the statement "$p$ and $q$" and is also a proposition.

**Example 2:** Today is Wednesday and it is sunny.
It is hot and I don't like it.

What is the truth value of a conjunction? Let's use "Today is Wednesday and it is sunny." as our example. Obviously, this is only true if it *is* Wednesday and it *is* sunny. In other words, "$p$ and $q$" is true if and only if both $p$ and $q$ are both true individually. A clear and easy way to discuss and/or determine the truth value of propositions is by using a *truth table*. A truth table is simply a concise way to list out all possibilities of truth value for the base propositions, and then determine the truth value of more complicated propositions using those. For example, the truth values of the conjunction are given in Table 1.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

**Table 1**: Truth Table for $p \wedge q$

**Definition:** The **disjunction** of $p$ and $q$, denoted by $p \vee q$, is the statement "$p$ or $q$" and is also a proposition. Table 2 gives the truth table.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**Table 2**: Truth Table for $p \vee q$

Notice that for $p \vee q$ to be true, all that is needed is for either $p$ or $q$ to be true (or both). This brings to mind an important distinction that needs to be made. In the English language, the word "or" can be used inclusively ($p$ or $q$ or both) or exclusively (either $p$ or $q$ but not both). In mathematics, we usually (and always in this class) use the inclusive "or".

**Example 3:** Determine whether the "or" in each sentence is inclusive or exclusive from the context.

(a) I would like a girlfriend or someone to play chess with.
(b) I need to pay my rent or I will be evicted.
(c) With the breakfast special, you get toast or a biscuit.
(d) The number is greater than 3 or less than 3.
(e) To attend Heck University, you must have a 21 ACT or a 3.0 high school gpa.

Statement (a) uses the inclusive "or" since it would certainly be fine if my girlfriend could play chess. Similarly, the "or" in statement (e) is also inclusive (since a student could have a 21 ACT score *and* a 3.0 high school gpa). In the other three statements, the exclusive "or" is used. If you pay your rent, you won't get evicted, you can't get toast *and* a biscuit with the breakfast special, and no number is *both* greater than 3 and less than 3.

**Definition:** The **negation** of proposition $p$ is also a proposition, denoted by $\neg p$ (or just $-p$). It has the opposite truth value from $p$.

This also reminds me of a delicate situation. When we speak of opposite conditions or statements in the English language, we often mean from one extreme to another. For example, to some people the opposite of "It's freezing in here." might be something like "It's hot in here." In mathematics, however, the negation of a statement (or its opposite) is something else. The negation of "It's freezing in here." would be "It's not freezing in here." Therefore it might still be chilly, or warm, or downright hot, just not freezing. While talking and thinking like this will be greatly beneficial in this class, I caution you *not* to do so out in the world. Literal-ness like that seems to infuriate people. Or so I've heard.

**Definition:** The **implication** (or **conditional** statement) "if $p$, then $q$", denoted $p \to q$, is also a proposition. In an implication, proposition $p$ is called the **hypothesis** and proposition $q$ is called the **conclusion**.

Implications are very common and can be written in many forms. The following are all equivalent:

If $p$, then $q$.
If $p$, $q$.

$p$ implies $q$.
$p$ only if $q$.
$q$ if $p$.
$p$ is sufficient for $q$.
$q$ is necessary for $p$.

What about the truth value of an implication? Consider the statement "If you score 90% or higher on the test, then you will receive an A." To determine when this statement is true, ask yourself, under what conditions would you feel justified claiming it was false?

(a) If you scored 90% or better and got an A? Looks true.
(b) If you scored 90% or better but did not get an A? Then it is false.
(c) If you did not score 90% or better, you wouldn't be able to claim it was false since you didn't even meet the condition of the hypothesis.

So an implication $p \rightarrow q$ is true unless $p$ is true and $q$ is false. See Table 3.

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Table 3**: Truth Table for $p \rightarrow q$

This reminds me of a very common technique in debating and politics. You can claim anything you want as a conclusion and still be telling the truth if the hypothesis was false. With false assumptions, you can state anything and still be honest. To see an example of this, consider the following scene.

**Example 4:** (Scene: Two lawyers argue a case.)

| | |
|---|---|
| Prosecutor: | If the defendant is guilty, then he acted alone. |
| Defense Attorney: | I object! That's a lie! |
| Judge: | Are you sure? |
| Defense Attorney: | Absolutely, it's not true! |
| Judge: | Very well. Then I declare a mistrial. The defendant needs a new attorney. |

What happened? Since the defense attorney claimed that the prosecutor had lied, he was claiming that the statement the prosecutor said was false. But the only way his statement (an implication) could be false is if the hypothesis was true and the conclusion was false. Therefore, the hypothesis had to be true. This meant that the defendant's attorney believed that the defendant was guilty and was saying so in court! Not a good idea.

There are a few other compound statements related to the basic implication.

**Definition:** The **converse** of the implication $p \to q$ is the proposition $q \to p$, the **contrapositive** is $-q \to -p$, and the **biconditional** is $p \leftrightarrow q$.

The biconditional can be read as "If $p$, then $q$ and if $q$, then $p$", "$p$ if and only if $q$", or "$p$ is necessary and sufficient for $q$." The truth values for the converse, the contrapositive and the biconditional are given in the following table.

| $p$ | $q$ | $p \to q$ | $q \to p$ | $-p$ | $-q$ | $-q \to -p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | T |
| T | F | F | T | F | T | F | F |
| F | T | T | F | T | F | T | F |
| F | F | T | T | T | T | T | T |

**Table 4**: Truth Table for $q \to p$, $-q \to -p$, and $p \leftrightarrow q$

It is often helpful to remember the truth values of these propositions. You don't want to have to repeatedly construct truth tables for these basic expressions. Notice that the truth value for $-q \to -p$ is identical to the truth value for $p \to q$. In terminology, this means that $p \to q$ and $-q \to -p$ are ***logically equivalent***. We will denote this fact by the symbol $(p \to q) \Leftrightarrow (-q \to -p)$. There are many common equivalent propositions that you will become familiar with. In proving theorems, it will sometimes be difficult (impossible even) to see how to prove the statement as given. Instead, proving an equivalent statement will be more reasonable.

**Example 5:** Determine the truth table for $-(-p \wedge (q \vee r))$.

| $p$ | $q$ | $r$ | $q \vee r$ | $-p$ | $-p \wedge (q \vee r)$ | $-(-p \wedge (q \vee r))$ |
|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T |
| T | T | F | T | F | F | T |
| T | F | T | T | F | F | T |
| T | F | F | F | F | F | T |
| F | T | T | T | T | T | F |
| F | T | F | T | T | T | F |
| F | F | T | T | T | T | F |
| F | F | F | F | T | F | T |

**Table 5**: Truth Table for $-(-p \wedge (q \vee r))$

I guess it is about time that I carefully explain how I generated this table, especially now that it is getting trickier. We first need to have every possible combination of truth values for the base propositions (variables). These are the first three columns. Then we gradually build to the pieces we need to evaluate in the complex proposition under consideration. In other words, to determine the truth value of $-(-p \wedge (q \vee r))$ (the negation of $-p \wedge (q \vee r)$), we just need to truth value of $-p \wedge (q \vee r)$. This is a conjunction, so we need the truth values of the pieces, namely $-p$ and $(q \vee r)$. This can be easily determined from the first three columns.

**Example 6:** Determine the truth table for $p \vee (-q \wedge -r)$.

| $p$ | $q$ | $r$ | $-q$ | $-r$ | $(-q \wedge -r)$ | $p \vee (-q \wedge -r)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T |
| T | T | F | F | T | F | T |
| T | F | T | T | F | F | T |
| T | F | F | T | T | T | T |
| F | T | T | F | F | F | F |
| F | T | F | F | T | F | F |
| F | F | T | T | F | F | F |
| F | F | F | T | T | T | T |

**Table 6**: Truth Table for $p \vee (-q \wedge -r)$

Notice that $-(-p \wedge (q \vee r)) \Leftrightarrow (p \vee (-q \wedge -r))$, i.e. the two propositions from the last two examples are equivalent. Since these two propositions are equivalent, the biconditional statement $-(-p \wedge (q \vee r)) \leftrightarrow (p \vee (-q \wedge -r))$ is always true. This is fairly common, and therefore has a name.

**Definition:** A statement that is always true is called a **tautology**. One that is always false is called a **contradiction**.

**Example 7:** The proposition $p \to p$ is a tautology and the statement $p \leftrightarrow -p$ is a contradiction for obvious reasons.